On the foundations of modern logic: Logical consequence and closure spaces

Luís M. Augusto

Material prepared as a visiting researcher at the **Department of Logic, History, and Philosophy of Science** of the **University of Barcelona**

Comments welcome: luis.ml.augusto at gmail.com

Abstract

The notion of logical consequence is at the very heart of modern logic. Indeed, a logical system S can be seen as an ordered pair $(\mathscr{L}, Cq)$ where $\mathscr{L}$ is a formal language and Cq is a consequence operation/relation on/in $\mathscr{L}$. Following Tarski (1930), one can approach Cq from the viewpoint of closure spaces: through this perspective, the theory of logical consequence can be seen as *fundamentally* a branch of applied topology.

Outline

1. Motivation
   - Centrality & nature of logical consequence
2. Logical consequence and closure spaces
   - Closure spaces and topology
   - Closure spaces and deduction
   - Main results
3. Foundational issues

A central notion, in logic and beyond

- "We must not lose sight of the fact that the concept of consequence is far more important than that of logical truth, both intuitively and technically. ... Where the notion of logical truth gains its importance is as the limiting case of the consequence relation: there are sentences that follow logically from any set of sentences whatsoever. The crucial notion, ultimately, is that of one sentence following logically from others. Logic is not the study of a body of trivial truths; it is the study of the relation that makes deductive reasoning possible." (Etchemendy, 1999, p. 11)
- "The crux of the matter is ... the definition of the term 'logical consequence'. Until this term has been explained, one does not have an opinion as to the nature of mathematics at all." (Curry, 1963, p. 10)

What is logical consequence?

- Logical consequence appears to be strongly connected to–perhaps inseparable from–other central notions in modern logic such as **derivability/deducibility/provability**, **validity**, and **consistency**.
- In effect, given a (possibly empty) set of formulae Γ and some formula ϕ, we say that ϕ is a syntactic consequence of Γ (denoted by $\Gamma \vdash \phi$) or a semantic consequence of Γ (denoted as $\Gamma \models \phi$) if *there is a proof of* ϕ from Γ, or if Γ *validates* ϕ, respectively.
- In turn, Γ is said to be *consistent* if at most one of ϕ, $\neg\phi$ is a consequence of Γ.

Some issues

- This, however, raises more questions than it answers from the viewpoints of the philosophy of logic and of philosophical logic:
  - To begin with, for some authors the very theoretical definition of logical consequence calls for a justification (e.g., McKeon, 2010).
  - In particular, an identification (conflation?) of logical consequence with either derivability or validity should be cautious at best (Etchemendy, 1999), or perhaps even simply avoided .
- By approaching logical consequence via closure spaces, it appears that both proof theory and model theory become irrelevant for this central notion, freeing it from questions of syntax and meaning.

Closure spaces

A closure space is a pair $\mathbb{C} = (S, \overline{\cdot})$ where $S \neq \emptyset$ and $\overline{\cdot} : 2^S \longrightarrow 2^S$ is a closure operator on S characterized by extensivity (C1), idempotence (C2), and isotonicity (C3):

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(C1) $A \subseteq \overline{A}$
(C2) $\overline{(\overline{A})} = \overline{A}$
(C3) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$ [equivalently: $\overline{A} \subseteq \overline{A \cup B}$]

(C1)-(C3) are jointly equivalent to the condition

☛

(C0) $A \subseteq \overline{B}$ iff $\overline{A} \subseteq \overline{B}$

Topological spaces

We define the closure of A as the smallest closed set in S containing A, i.e.,

$$\overline{A} = \bigcap_{C \supseteq A, closed} C$$

If C is seen as belonging to a given family of closed sets $\mathscr{C}$, then $\bar{\cdot}$ is a hull operator satisfying three (K1, K2, and K3*) of the Kuratowski closure axioms that define a topological space $\mathbb{S} = (S, \mathscr{T})$ for $A, B \in 2^S$:

☛

(K1) $A \subseteq \overline{A}$

(K2) $\overline{(\overline{A})} = \overline{A}$,

(K3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(K3*) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$

(K4) $\overline{\emptyset} = \emptyset$

The Tarskian axioms

Tarski (1930): Let S be the set of **all** *meaningful sentences* (i.e., wffs) and let A be an arbitrary set of sentences. With the help of certain operations called *rules of inference* new sentences are derived from A; these sentences are called *the consequences of the set* A. For $|S| \leq \aleph_0$ [(A*)], if $A, B \subseteq S$, then

☛

(A1) $A \subseteq Cq(A)$
(A2) $Cq(Cq(A)) = Cq(A)$
(A3) If $A \subseteq B$, then $Cq(A) \subseteq Cq(B)$

(A1)-(A3) amount to that for all $A, B \subseteq S$,

☛

(A0) $A \subseteq Cq(Cq(A)) \subseteq Cq(A) \subseteq Cq(A \cup B)$

The Tarski space

- Let $Cq : 2^S \longrightarrow 2^S$ be the closure operator that assigns to each $A \in 2^S$ a subset $Cq(A) = \overline{A}$ of S, and define a subset A of S to be closed iff
$$Cq(A) = A$$
Then, each closure operator Cq determines a unique topology $\mathscr{T} = \{A \subset S : S - A \, \mathrm{is\, closed}\}$ on S.
- In this way, we have constructed a *Tarski space* $\mathbb{T} = (S, Cq)$ from the topological space $\mathbb{S}$ s.t. the axioms of the closure space $\mathbb{C} = (S, \bar{\cdot})$ apply / are satisfied.
- We have thus made of logical consequence an object of topology.

Deductive closure

- Now let $\vdash$ be a derivability relation defined on a set of sentences $\mathscr{L}$. For $\Gamma \subset \mathscr{L}$ and $\phi \in \mathscr{L}$, if $\Gamma \vdash \phi$, we say that ϕ is derivable from Γ, and the *deductive closure* of Γ is the set $\overline{\Gamma} = \{\phi \in \mathscr{L} : \Gamma \vdash \phi\}$.
- Then, (C1)-(C3) have a "deductive closure interpretation":

	Deductive closure
(C1)	$\Gamma \vdash \phi$ whenever $\phi \in \Gamma$
(C2)	$\overline{\Gamma} \vdash \phi$ whenever $\phi \in \Gamma$
(C3)	$(\Gamma \cup \Gamma^*) \vdash \phi$ whenever $\Gamma \vdash \phi$

Cq is a topology

Building up from (1) the Tarski space, (2) further "correspondences" between logic and topology (e.g., denseness and inconsistency; neighborhood and non-theoremhood, homeomorphism and equivalence), and (3) other topological notions such as interior, boundary, etc.,

- (α) *a substantial part (the whole?) of the theory of (Tarskian) logical consequence can be formulated within topology* (e.g., Martin & Pollard, 1996)*;*
- (β) *many features of classical and non-classical logics find a "natural" topological expression/interpretation* (*ibid.*).

Some examples follow.

Some (not so) elementary theorems

For $A, B \subset S$ given $\mathbb{C} = (S, \bar{\cdot})$,

- If B is closed and $A \subset B$, then $\overline{A} \subset B$
- $\overline{A \cup B} = \overline{A}$ iff $B \subset \overline{A}$
- $\overline{A \cup \overline{\emptyset}} = \overline{A}$
- $\overline{\bigcap_\alpha A_\alpha} \subset \bigcap_\alpha \overline{A_\alpha}$
- $\bigcup_\alpha \overline{A_\alpha} \subset \overline{\bigcup_\alpha A_\alpha}$
- Each maximally consistent set is closed.
- $\overline{A} = A \cup A'$, where A' is the derived set of A
- If A is closed, then so is A'

Some further results

- $A \models x$ iff $x \in \overline{A}$
- Let S^* be a set of truth values and $v : S \longrightarrow S^*$. Suppose further that $T \subset S^*$. If $\neg$ is a classical negation and v is a correct valuation of S, then $\forall x \in S$, $v(\neg x) \in T$ iff $v(x) \notin T$.
- Let $\dot{A} = S - \overline{S - A}$, i.e., $\dot{A}$ denotes the interior of A. Let now $\Diamond A$ stand for $\overline{A}$, $\Box A$ for $\dot{A}$, and let $\sim A$ denote $S - A$. Then we have that

  $$\Box A = \sim \Diamond \sim A$$

  and all of the modal logic S4 can be constructed within $\mathbb{C}$, i.e., S4 has a topological interpretation (formulation?).

Logic and topology

- "... over a large body of logic, the closure structure is basically its essence and its syntactic and linguistic features are secondary, useful for purposes of understanding though they may be." (Martin & Pollard, 1996, p. xiv)
- "... the abstract theory of derivability and consequence is fundamentally a branch of applied topology." (*ibid.*, p. xiii)

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